International Journal of Theoretical Physics, Vol. 8, No. 4 (1973), pp. 231–235

Smoothing the Quantum Collapse

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Received: 14 *December* 1972

Abstract

A generalisation of quantum theory is presented in which the state of the universe is represented by a vector which varies smoothly with time. In the presence of measurements a smoothed-out 'collapse' takes place, while in their absence conventional Hamiltonian dynamics are obtained.

1. *Introduction*

In this note we present one way in which quantum theory can be 'universalised' into a form suitable for cosmological applications, restricting the discussion to a brief presentation of the formalism. A comparison with the Wheeler–Everett interpretation together with a consideration of the objections to 'collapse' and of the broader issues involved will be published elsewhere (Clarke, 1973).

It will be shown that it is possible to generalise quantum theory (in the sense that an orthodox quantum theory of small systems is contained as a special case) into a complete theory with the following properties.

I. There is a cosmic time t such that for each value of t the universe has a definite 'state' described by a vector $\Phi(t) \in V$, where V is a Banach space.
II. $\Phi(t)$ is a continuous function of t.
III. If $t_2 - t_1 = t > 0$ then $\Phi(t_2)$ is determined by a conditional probability measure $P_t(A|\Phi(t_1))$, where, for each $\Phi \in V$, $P_t(.|\Phi)$ is a probability measure on the σ-algebra $\mathscr{B}_V$ generated by the cylinder-sets of V.
IV. If $P_t(.|\Phi)$ for $t > 0$ is absolutely continuous with respect to a fixed standard measure μ, then its Radon–Nikodym derivative q satisfies

$$\frac{dq}{dt} = Lq$$

where L is a second-order differential operator depending only on Φ.

V. $\lim_{t \to 0} P_t(A|\Phi) = \chi_A(\Phi)$.

Remark. The interpretation III may give rise to difficulties, since the 'probability' of the universe having a certain state is not a testable quantity: probability statements should be confined to ensembles within the universe. It is shown in Clarke (1973), by adopting an idea due to Everett (Everett, 1957) that in an infinite universe it is sufficient to take:

III′. The interpretation of P is as follows. If at time t_1 the universe is in a state Φ_1, then a class of states $A \in \mathscr{B}_V$ with $\mu(A) \neq 0$ is to be regarded as *possible* at time $t_1 + t$ if, and only if, $P_t(A|\Phi_1) \neq 0$.

However, in an infinite universe it is not clear that the equation in IV above has any solutions, and so we only consider the finite case.

2. *General Ideas*

We construct the theory by adopting the 'collapse of the wave-packet' doctrine in its most naive form, and supplying an explicit description of how the wave-packet collapses. We require, as in II above, that this description be continuous, so that we assign a characteristic time-scale τ_0 to a particular collapse.

Continuity is adopted because in cosmology we cannot hope for a neat division of all physical situations at all times into those which can be called 'measurements with macroscopic apparatus' and those which cannot be so designated. We thereby allow a continuous gradation between 'classical measurements' (τ_0 very small) and 'non-measurement interactions' (τ_0 very large, no collapse takes place).

The theory is incomplete as far as its explicit statement is concerned, in that no criterion is known at present for this 'measurement-ness' of a situation. Thus there appears an unspecified function $S(\Phi)$ which defines for any given state of the universe Φ, what 'measurements', in a general sense, are going on and what the τ_0 is that is associated with each. It is reasonable to expect that S can be defined, because the formalism itself ensures that Φ is (almost) always interpretable as a definite macroscopic state and (almost) never describes a superposition of macroscopically distinguishable states.

The lack of a criterion as to what is and what is not a measurement is an essential problem in all attempts to generalise quantum theory to a cosmological setting, a problem which we in no way solve. What is done is to pave the way for the incorporation of a quantitatively graded solution, instead of demanding a 'yes' or 'no' answer. As has been shown in Clarke (1973) the necessity for such a graded solution only arises in theories which satisfy I above. The Wheeler–Everett (Everett, 1957) interpretation does not fall into this category, and so does not have to incorporate the sort of collapse mechanism we shall describe.

3. *Smoothing the Collapse*

We start from a conventional quantum system, with a Hilbert space $\mathscr{H}$ and Hamiltonian H, and look for a natural way of describing how a measurement causing a collapse could be spread out over an interval of time τ_0. One possibility can immediately be dismissed: that of taking repeated measurements at times τ_0/N, $2\tau_0/N$, . . ., τ_0 and letting $N \to \infty$, for the first measurement produces a total collapse and subsequent ones serve only to maintain the state in an eigenvector. But we could apply the idea if each measurement produced something less than a total collapse, which can indeed be achieved by using 'mixed measurements' (Giles, 1970).

Originally this concept was introduced in a 'quantum logic' type of setting, differing radically from ours. Thus we borrow the formalism only and endow it with a somewhat non-standard interpretation. First we describe Giles's original idea.

We recall that a *mixed state* Φ is represented by a compact Hermitian operator of unit trace; V will be defined as the set of all compact Hermitian operators of trace class. Thus $\Phi(\varphi) = \sum_a \psi_a(\psi_a, \varphi)p_a$ or $\Phi = \sum_a p_a[\psi_a]$, where $[\psi]$ denotes the projection on ψ. We normalise through $(\psi_a, \psi_a) = \sum_a p_a = 1$. One could interpret Φ as an expression of the knowledge that the state ψ_a is present with probability p_a for all a. A state of the form $[\psi]$ is a *pure state*.

A *test* is a special case of a measurement which can only yield the outcomes 0 or 1. Thus it is represented by a projection operator P, and when Φ is measured the result (3.1) is obtained with

$$(\text{probability of } 1) = \sum p_a(\psi_a, P\psi_a) = \mathrm{Tr}\, P\Phi$$

Now consider the example (due to Giles) of a photon-counter set up to count right-polarised photons, so realising the measurement described by a projection P_r. In reality a fraction ε of right-polarised photons will always escape undetected, while a fraction δ of left-polarised ones will accidentally trigger the counter, so that the probability of response to a state Φ is actually $p = (1-\varepsilon)\mathrm{Tr} P_r \Phi + \delta \mathrm{Tr} P_l \Phi$ where P_l is the left-photon projection. We define R to be $(1-\varepsilon)P_r + \delta P_l$ so that $p = \mathrm{Tr} R\Phi$: R is called a *mixed test*, and describes an imperfect measurement. In general we define a mixed test R to be a Hermitian operator with spectrum in $[0, 1]$, interpreted by setting

$$\text{probability of response in state } \Phi = \mathrm{Tr}\, R\Phi \tag{3.1}$$

A collection $\mathbf{R} = \{R^\alpha | \alpha = 1, 2, \ldots, n\}$ of commuting mixed tests is called a *mixed measurement*. We are here splitting a measurement with a finite set of outcomes into a sequence of yes/no tests. The outcome of such a measurement is a sequence $\mathbf{h} = (h^\alpha | \alpha = 1, \ldots, n)$ with $h^\alpha \in \{0, 1\}$.

We now interpret this concept to apply in the situation of I, producing a collapse. Suppose that one mixed measurement is followed by another, so that the first can be regarded as a 'preparation' for the second. Then the results will be consistent if the first measurement is regarded as producing the collapse

$$\Phi \to R'^{1/2}\, \Phi R'^{1/2}/\mathrm{Tr}\, R'\Phi \tag{3.2}$$

where

$$R'^{1/2} = \prod_{\alpha=1}^{n} \sum_{j=1}^{m_\alpha} (a_j'^{\alpha})^{1/2} P_j^{\alpha}$$

the R'^{α} having spectral resolutions†

$$R'^{\alpha} = \sum_{j=1}^{m_\alpha} a_j'^{\alpha} P_j^{\alpha} \tag{3.3}$$

and being defined by $R'^{\alpha} = (2h^{\alpha} - 1)R^{\alpha} + (1 - h^{\alpha})I$ so that R'^{α} is the experiment from the pair $(R^{\alpha}, I - R^{\alpha})$, which results in success.

Note that, from some conventional points of view, this might be regarded as being partly a quantum collapse and partly an increase in information.

In the limit when **R** is a pure test ($\varepsilon = \delta = 0$ in the example) this is the conventional collapse in 'density matrix' form, while in the limit $R^{\alpha} \to I$, when the measurement yields no information at all ($\varepsilon = \delta = \frac{1}{2}$ in the example), the state is unchanged. Thus **R** meets our requirements for a partial collapse, and we can use it to construct a smoothed collapse in the way already indicated.

Define

$$R_\tau^{\alpha} = \sum_{j=1}^{m_\alpha} \left[a_j^{\alpha} \left(\frac{\tau}{\tau_0}\right)^{1/2} + \frac{1}{2}\left(1 - \left(\frac{\tau}{\tau_0}\right)^{1/2}\right)\right] P_j^{\alpha}$$

where the spectral resolution of R^{α} is defined similarly to (3.3). Perform measurements with the mixed measurement $\mathbf{R}_\tau = \{R_\tau^{\alpha} | \alpha = 1, \ldots, n\}$ at intervals of τ_0/N over a time-span t. The eventual state after this span depends on the results of the measurements which, *ex hypothesi*, are assigned probabilities by (3.1), so that we obtain a probability distribution for the final state. In effect, the state Φ executes a random walk, taking one step at each measurement. The distribution has a limit ν as $N \to \infty$, and the usual theory of random walks shows that if this limit is suitably smooth it satisfies

$$d\nu/dt = K_{\mathbf{R}}\, \nu \stackrel{\text{def}}{=} \tfrac{1}{2} \sum_{\alpha=1}^{n} \mathscr{L}_{X^\alpha} \mathscr{L}_{X^\alpha} \nu - 2 \sum_{\alpha=1}^{n} \mathscr{L}_{X^\alpha} c^{\alpha} \nu$$

where $\mathscr{L}_X$ is the Lie derivative with respect to X, defined by

$$(\mathscr{L}_X \rho)(A) = \lim_{b \to 0} ((1/t)(\rho(A) - \rho(\varphi_t^{-1}(A)))$$

with φ_t a 1-parameter family of diffeomorphisms generating X, and we set

$$X^{\alpha} = [S^{\alpha}, \Phi]_+|_{\Phi}, \qquad c^{\alpha} = \mathrm{Tr}\, \Phi S^{\alpha}/\mathrm{Tr}\, \Phi, \qquad S^{\alpha} = 2R_1^{\alpha} - I$$

† We shall assume that the R^{α} have discrete spectra, and generalise to continuous spectra at the end.

writing $A|_B \in T_B(V)$ for the tangent vector at B parallel to A when $A, B \in V$. In deriving this we have dropped the normalisation condition $\mathrm{Tr}\,\Phi = 1$. In the limit $\tau_0 \to 0$ we obtain the usual collapse, at least in the finite dimensional case.

We can incorporate the Hamiltonian evolution on recalling that in the absence of measurements

$$\Phi(t+\tau) = \Phi(t) + [H, \Phi(t)]\,\tau/i\hbar + O(\tau^2)$$

In terms of the present picture, the Hamiltonian supplies a systematic drift of the probability distribution described by the vector $Z = [H, \Phi(t)]/i\hbar|_\Phi$ which is added to the random-walk behaviour. This drift is responsible for the usual quantum phenomena (interference and so on), while the random walk describes measurements.† As a result, we take for the equation of motion

$$\frac{d}{dt}P_t(.\,|\,\Phi) = [K_{\frac{1}{2}(\mathbf{I}+\mathbf{S})} - \mathscr{L}_Z]\,P_t(.\,|\,\Phi_1) \tag{3.4}$$

Here, as already described, we suppose that $\mathbf{R}$ includes all the measurements and is specified by giving $\mathbf{S}$ as a function of Φ.

The restrictions on the spectra of the R^α and the finite range of α can now be relaxed provided that the right-hand side of (3.4) remains defined.

References

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† The reactions of some critics make it necessary to stress that I am not replacing the wave-function by a probability distribution over the values of observables, which would, of course, make 'interference' impossible. In fact, in the limit $\tau_0 \to \infty$ (no measurements), ν becomes a δ-function whose support describes precisely the path predicted by conventional quantum mechanics.